

**Quantum multiple scattering: Eigenmode expansion and its applications to proximity resonance**

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We show that for a general system of  $N$   $s$ -wave point scatterers, there are always  $N$  eigenmodes. These eigenmodes or eigenchannels play the same role as spherical harmonics for a spherically symmetric target—they give a phase shift only. In other words, the  $T$  matrix of the system is of rank  $N$ , and the eigenmodes are eigenvectors corresponding to nonzero eigenvalues of the  $T$  matrix. The eigenmode expansion approach can give insight to the total scattering cross section; the position, width, and superradiant or subradiant nature of resonance peaks; the unsymmetric Fano line shape of sharp proximity resonance peaks based on the high-energy tail of a broadband; and other properties. Off-resonant eigenmodes for identical proximate scatterers are approximately angular-momentum eigenstates.

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**I. INTRODUCTION**

Scattering of waves from a group of scatterers is rich in interesting physical phenomena. Examples are electrons or phonons scattering from defects or impurities in crystals and light scattering from conjugated molecules or biological complexes.

Multiple scattering effects are nontrivial, especially when scatterers are placed close, i.e., inside each other's effective radius  $\sqrt{\sigma/\pi}$ , where  $\sigma$  is the total cross section of a single scatterer. If the scatterers are resonant, their effective radius can be much larger than their physical size or force range. A classical example is the case of weak, fixed frequency sound incident on two proximate, small identical air bubbles in water [1–3]. Quantum mechanically, for two scatterers placed together well within the resonant wavelength of scattered particle, extremely narrow proximity resonance can appear [4,5]. Three or more proximate scatterers lead to related effects.

In this paper we have formulated a general method, eigenmode expansion, for quantum scattering of a system of  $s$ -wave point scatterers of any number and geometric configuration. The eigenmode expansion is based on the Green's function method as described in Refs. [10,15]. The Green's function method is equivalent to solving the Lippmann-Schwinger equation exactly—taking all orders of multiple scattering into account.

It is obvious from eigenmode expansion approach that the  $T$  matrix of such a system is of rank  $N$  and we are able to construct the  $T$  matrix explicitly. Many features of the spectrum (total scattering cross section as a function of energy) of identical proximate scatterers can be explained simply by this approach. For example, we can naturally explain the position, width, and superradiant or subradiant nature of resonance peaks, and the unsymmetric Fano line shape of

sharp proximity resonance peaks based on the high-energy tail of a broad band. We are also surprised to find that off-resonant eigenmodes of *randomly distributed* identical proximate scatterers exhibit great symmetry and regularity. Similar systems of  $N$  interacting resonances have been previously studied, e.g., Refs. [6,7,15]. Though our eigenmode expansion approach is of course equivalent to standard scattering theory, it gives new physical insights, as we demonstrate below.

In addition, the quantum scattering of  $N$  proximate  $s$ -wave scatterers is related to the superradiance problem [8,9]. Again, our eigenmode expansion gives new insights to superradiance.

**II. REVIEW OF QUANTUM MULTIPLE SCATTERING**

We start by introducing our model for  $s$ -wave point scatterers. Consider a single elastic scatterer placed at  $\vec{r}=0$ . The asymptotic form of the total wave function is

$$\psi_{\vec{k}}(\vec{r}) \cong e^{i\vec{k}\cdot\vec{r}} + f(\vec{k}, \hat{r}) \frac{e^{ikr}}{r} \quad \text{for } r \rightarrow \infty. \quad (1)$$

The first term on the right-hand side is the incoming plane wave, and the second term is the scattered wave. The angle-dependent scattering amplitude  $f(\vec{k}, \hat{r})$  satisfies the optical theorem:  $(4\pi/k) \text{Im} f(\vec{k}, \hat{r}=\hat{k}) = \sigma(\vec{k})$ , where the total cross section  $\sigma(\vec{k}) \equiv \int |f(\vec{k}, \hat{r})|^2 d\hat{r}$ . The  $T$  matrix of the scatterer in the subspace of energy shell  $|\vec{k}|=k$ ,  $T_k$ , can be defined as an operator acting on functions of  $\hat{r}$ :

$$T_k y(\hat{r}) \equiv \frac{k}{4\pi} \int f(k\hat{r}', \hat{r}) y(\hat{r}') d\hat{r}'.$$

We assume the force range of the scatterer is small compared to the wavelength of the scattered particle, so it only scatters the  $s$ -wave component  $[j_0(kr)]$  of the incoming wave. The scattering amplitude  $f(k)$  is now angle-independent, and is

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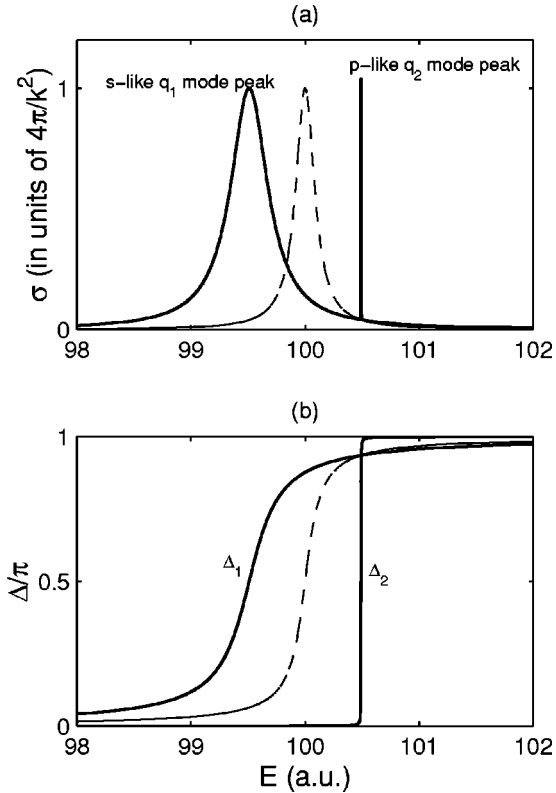


FIG. 1. Solid lines: (a) total cross section  $\sigma$  and (b) phase angles  $\Delta_j$  of each eigenmode as a function of energy of a system of two identical scatterers separated by distance  $d=0.2/k_0$ . Each scatterer is specified by Eq. (2) with  $E_0=100$  a.u. and  $\gamma_0=0.1$ . For comparison, the total cross section and phase angle of a single scatterer are also shown (dashed lines).

related to the phase angle  $\delta(k)$  by  $f(k)=(1/k)\sin\delta(k)e^{i\delta(k)}$ . The  $T$  matrix is of rank 1.  $T_k$  has only one nonzero eigenvalue  $\sin\delta(k)e^{i\delta(k)}$ , whose corresponding eigenvector is the zeroth-order spherical harmonic.

A simple scatterer with one internal Breit-Wigner-type resonance can be modeled as

$$f(k) = -\frac{1}{k} \frac{\gamma_0}{k^2 - \frac{k_0^2}{2} + i\gamma_0}. \quad (2)$$

The total cross section has a Lorentzian shape [dashed line in Fig. 1(a)]:

$$\sigma(E) = \frac{4\pi}{k^2} \frac{\gamma_0^2}{(E - E_0)^2 + \gamma_0^2},$$

where  $E=k^2/2$  and  $E_0=k_0^2/2$ . Resonance corresponds to the pole of  $f(E)$  on complex  $E$  plane, at  $E_0 - i\gamma_0$ , where the “scattered wave” can exist without the incoming wave. Physically,  $E_0$  and  $\gamma_0$  correspond to position and half-width of the resonance peak.

We further assume that the force range of the scatterer is so small that it can be considered as existing only at one point. In this point scatterer model, Eq. (1) holds not only

asymptotically, but over all space. Since any incoming wave  $\phi(\vec{r})$  of energy  $E=k^2/2$  can be written as a superposition of plane waves with  $|\vec{k}|=k$ , the total wave function can be written as

$$\psi(\vec{r}) = \phi(\vec{r}) + \phi(0)f(k)G_k(r), \quad (3)$$

where the free space Green’s function  $G_k(r)=e^{ikr}/r$ . The only significant difference to Eq. (1) is the extra factor  $\phi(0)$ , the amplitude of the incoming wave at the point scatterer.

We are now ready to tackle the multiple-scatterer problem using the Green’s function method described in Refs. [10,15]. The total wave function for a system of  $N$  scatterers fixed at positions  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$  is

$$\psi(\vec{r}) = \phi(\vec{r}) + \sum_{i=1}^N \psi_i(\vec{r}_i)f_iG_i(\vec{r}), \quad (4)$$

where  $f_i$  is the scattering amplitude of the  $i$ th scatterer,  $G_i(\vec{r})$  is the free space Green’s function from  $\vec{r}_i$ :  $G_i(\vec{r}) = e^{ik|\vec{r}-\vec{r}_i|}/|\vec{r}-\vec{r}_i|$ , and for simplicity we have omitted all implicit  $k$  (energy) dependences. The  $\phi(0)$  in Eq. (3) is replaced by the amplitude at the  $i$ th scatterer of the  $i$ th effective incoming wave  $\psi_i(\vec{r})$ —the sum of the incoming wave and waves scattered by all scatterers except by the  $i$ th scatterer itself:

$$\psi_i(\vec{r}) = \phi(\vec{r}) + \sum_{j \neq i} \psi_j(\vec{r}_j)f_jG_j(\vec{r}). \quad (5)$$

The above equation provides a system of linear equations for  $\psi_i(\vec{r}_i)$ . We can express them explicitly in matrix form. Define vector for incoming wave  $\vec{\phi}$  and vector for effective incoming waves  $\vec{\psi}$  as  $\phi_i \equiv \phi(\vec{r}_i)$ ,  $\psi_i \equiv \psi_i(\vec{r}_i)$  ( $i=1, \dots, N$ ).

Define the  $N \times N$  free space Green’s matrix  $\mathbf{G}$  as

$$G_{ij} \equiv \begin{cases} G_j(\vec{r}_i) & \text{for } i \neq j, \\ 0 & \text{for } i = j \end{cases}$$

and matrix  $\mathbf{F}$  as

$$\mathbf{F} \equiv \text{diag}\{f_1, f_2, \dots, f_N\}.$$

By substituting  $\vec{r}$  with  $\vec{r}_i$  in Eq. (5), we get

$$\vec{\phi} = \mathbf{M}\vec{\psi}, \text{ or } \vec{\psi} = \mathbf{M}^{-1}\vec{\phi}, \quad (6)$$

where

$$\mathbf{M} \equiv \mathbf{1} - \mathbf{GF}.$$

The physical implication of Eq. (6) is clear. Formally,

$$\mathbf{M}^{-1} = \mathbf{1} + \mathbf{GF} + (\mathbf{GF})^2 + \dots,$$

where each term  $(\mathbf{GF})^l$  ( $l=0,1,\dots$ ) means the incoming wave is scattered  $l$  times, since  $\mathbf{F}$  means scattering once by

one scatterer and  $\mathbf{G}$  means free propagation from one scatterer to another. An integral equation similar to Eq. (6) is known in general scattering theory as the Lippmann-Schwinger equation [11]. Resonances occur at poles of  $\mathbf{M}^{-1}(E)$  on complex  $E$  plane.

The scattered wave  $\phi_{sc}(\vec{r})$  depends only on  $\vec{\phi}$ , the amplitudes of the incoming wave at scatterers. Asymptotically,

$$\phi_{sc} \equiv \left( \sum_{i=1}^N \psi_i f_i e^{-ik\hat{r}\cdot\vec{r}_i} \right) \frac{e^{ikr}}{r} \text{ for } r \rightarrow \infty. \quad (7)$$

If the incoming wave is a plane wave  $e^{i\vec{k}\cdot\vec{r}}$ , the sum in parentheses of Eq. (7) is nothing else but the scattering amplitude  $f(\vec{k}, \hat{r})$ .

### III. FORMULATION OF THE EIGENMODE EXPANSION

For a single scatterer, for incoming wave  $j_0(kr)$  ( $= [G(r) - G^*(r)]/2ik$ ), the scattered wave is proportional to  $G(r)$ —the outward part of the incoming wave—with a phase factor. Is there any incoming wave for multiple scatterers such that the scattered wave is proportional to the outward part of the incoming wave with a phase factor? This is equivalent to looking for eigenvectors of the  $T$  matrix of the whole system. In the spherically symmetric case, diagonalization of  $T$  matrix is essentially spherical harmonic partial-wave expansion. Some textbooks have discussed the formal diagonalization of the  $T$  (or  $S$ ) matrix for a general system, and its applications to scattering of particles with spin and scattering reactions [12]. We have found, as will be shown immediately, that for a system of  $s$ -wave point scatterers of any number and geometric configuration, the  $T$  matrix can be analytically represented under some special basis, and can be diagonalized easily.

From Eq. (5), we see the scattered wave of multiple scatterers is a superposition of  $G_i(\vec{r})$ 's, so we try to write the incoming wave as a superposition of  $j_0(k|\vec{r}-\vec{r}_i|)$ 's:

$$\phi(\vec{r}) = \sum_{i=1}^N q_{ij} j_0(k|\vec{r}-\vec{r}_i|) = \sum_{i=1}^N q_{ij} \frac{G_i(\vec{r}) - G_i^*(\vec{r})}{2ik}, \quad (8)$$

or  $\vec{\phi} = \mathbf{J}\vec{q}_j$ , where  $\vec{q}_j = (q_{1j}, \dots, q_{Nj})^T$ , and the  $N \times N$  spherical Bessel matrix  $\mathbf{J}$  is defined as

$$J_{ij} \equiv j_0(k|\vec{r}_i - \vec{r}_j|).$$

We have  $\vec{\psi} = \mathbf{M}^{-1}\vec{\phi}$ , and from Eq. (4), we get

$$\phi_{sc}(\vec{r}) = \sum_{i=1}^N G_i(\vec{r}) [\mathbf{F}\mathbf{M}^{-1}\mathbf{J}\vec{q}_j]_i = \frac{1}{k} \sum_{i=1}^N G_i(\vec{r}) [\mathbf{T}\mathbf{J}\vec{q}_j]_i,$$

where matrix  $\mathbf{T}$  is defined as

$$\mathbf{T} \equiv k\mathbf{F}\mathbf{M}^{-1}.$$

If  $\vec{q}_j$  is an eigenvector of  $\mathbf{T}\mathbf{J}$ ,

$$\mathbf{T}\mathbf{J}\vec{q}_j = \lambda_j \vec{q}_j (j = 1, \dots, N),$$

the scattered wave

$$\phi_{sc}(\vec{r}) = \frac{\lambda_j}{k} \sum_{i=1}^N G_i(\vec{r}) q_{ij}$$

is proportional to the outward part of the incoming wave. The angle dependence of the incoming and scattered wave is

$$y_j(\hat{r}) \equiv \sum_{i=1}^N q_{ij} e^{-ik\hat{r}\cdot\vec{r}_i}. \quad (9)$$

Each  $\vec{q}_j$  or  $y_j(\hat{r})$  can be regarded as an *eigenmode* or eigenchannel of the system.  $q_{ij}$  is the relative amplitude of the scattered wave from the  $i$ th scatterer. Eigenvalue  $\lambda_j$  is related to the phase angle of the  $j$ th eigenmode  $\Delta_j$  by

$$\lambda_j = \sin \Delta_j e^{i\Delta_j}.$$

Eigenmodes have some basic properties that can be proved straightforwardly.

(1)  $\vec{q}_j$  can always be chosen real due to time reversal symmetry,

$$q_{ij}^* = q_{ij}.$$

(2) The eigenmodes obey orthonormal relations upon normalization,

$$\vec{q}_i^T \mathbf{J} \vec{q}_j = \frac{1}{4\pi} \delta_{ij},$$

or

$$\int y_i^*(\hat{r}) y_j(\hat{r}) d\hat{r} = \delta_{ij}.$$

(3)  $\lambda_j$ 's and  $y_j(\hat{r})$ 's are actually eigenvalues and eigenvectors of the  $T$  matrix,

$$T y_j(\hat{r}) = \sin \Delta_j e^{i\Delta_j} y_j(\hat{r}).$$

It is clear now that  $y_j(\hat{r})$ 's here play the same role as spherical harmonics in spherically symmetric case. Spherical harmonics are eigenchannels under special conditions, namely, when the system is spherically symmetric.

Any incoming wave can be written as  $\phi(\vec{r}) = \sum_{i=1}^N c_i j_0(k|\vec{r}-\vec{r}_i|) + \phi'(\vec{r})$ , where  $\phi'(\vec{r})$  is orthogonal to each  $j_0(k|\vec{r}-\vec{r}_i|)$ . Immediately we see that the  $\phi'(\vec{r})$  part does not scatter at all, because each scatterer only scatters  $s$  wave, and  $\phi'(\vec{r})$  is orthogonal to each  $j_0(k|\vec{r}-\vec{r}_i|)$ . For a single scatterer, the  $T$  matrix is of rank 1. For a system of  $N$  scatterers, the  $T$  matrix is of rank  $N$ , since the system only scatters incoming waves belonging to the  $N$  dimensional function space the sum in Eq. (8) spans.

There is a simpler explanation for the rank of the  $T$  matrix. Recall from Sec. II that the scattered wave  $\phi_{sc}(\vec{r})$  depends only on  $\vec{\phi}$ , the amplitudes of the incoming wave at scatterers. For all the infinitely degenerate incoming waves of energy  $E = k^2/2$ , we can have a basis in which all but  $N$  waves have zero amplitude at all  $\vec{r}_i$ 's. Consequently, the sys-

tem only scatters the  $N$  waves that have nonzero amplitudes at  $\vec{r}_i$ 's, and does not scatter the rest at all.

In eigenmode basis  $y_j(\hat{r})$  [plus all other linearly independent functions of  $\hat{r}$  orthogonal to all  $y_j(\hat{r})$ 's], the  $T$  matrix has a diagonal form:

$$T = \text{diag}\{\sin \Delta_1 e^{i\Delta_1}, \dots, \sin \Delta_N e^{i\Delta_N}, 0, \dots\}.$$

The eigenmodes are actually eigenvectors corresponding to nonzero eigenvalues of the  $T$  matrix. In fact, matrix  $\mathbf{T}$  is the representation of  $T$  matrix in the (nonorthogonal) basis

$$b_j(\hat{r}) \equiv \frac{1}{\sqrt{4\pi}} \sum_{i=1}^N J_{ij} e^{-ik\hat{r} \cdot \vec{r}_i}$$

[or  $\mathbf{J}\mathbf{T}\mathbf{J}$  is the representation of  $T$  matrix in the  $(1/\sqrt{4\pi})e^{-ik\hat{r} \cdot \vec{r}_j}$  basis]. This can be verified if you view Eq. (9) as transformation of basis.

We can use eigenmode expansion to calculate the total cross section for an incoming plane wave. Optical theorem can also be verified. The total scattering cross section not only depends on the energy (or  $k$ ), but also the direction  $\hat{k}$  of the incoming wave. This is what we would expect for a system without spherical symmetry. The average of total cross section over all possible incoming directions for a given energy is independent of  $\hat{k}$ . From now on, we will use "total cross section" in this sense. The total cross section

$$\sigma = \frac{4\pi}{k^2} \sum_{j=1}^N \sin^2 \Delta_j$$

is a sum of contributions from each eigenmode [13].

Numerical evidence shows that for scatterers with the properties of Eq. (2), each phase angle  $\Delta_j$  monotonically increases from 0 to  $\pi$  as the energy goes from  $-\infty$  to  $\infty$ . Therefore each eigenmode will contribute a resonance peak to total cross section (if the peaks do not overlap). The position of the peak is at  $\Delta_j = \pi/2$ , and the width is determined by how fast  $\Delta_j$  passes through  $\pi/2$  and nearby region as a function of energy. For a system of  $N$  scatterers, there will be  $N$  resonance peaks in the total cross section, whose positions and widths are consistent with the  $N$  poles of  $\mathbf{M}^{-1}(E)$  on complex  $E$  plane.

#### IV. APPLICATIONS OF THE EIGENMODE EXPANSION TO THE PROXIMITY RESONANCE

First we consider the simplest case of multiple scattering: two identical scatterers separated by distance  $d$ . It is easy to verify that  $\mathbf{F}$ ,  $\mathbf{M}$ , and  $\mathbf{J}$  all have the form of

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

so does  $\mathbf{T}\mathbf{J} = k\mathbf{F}\mathbf{M}^{-1}\mathbf{J}$ . Thus without knowing  $f(k)$  explicitly, we can immediately say that the two (unnormalized) eigenmodes are

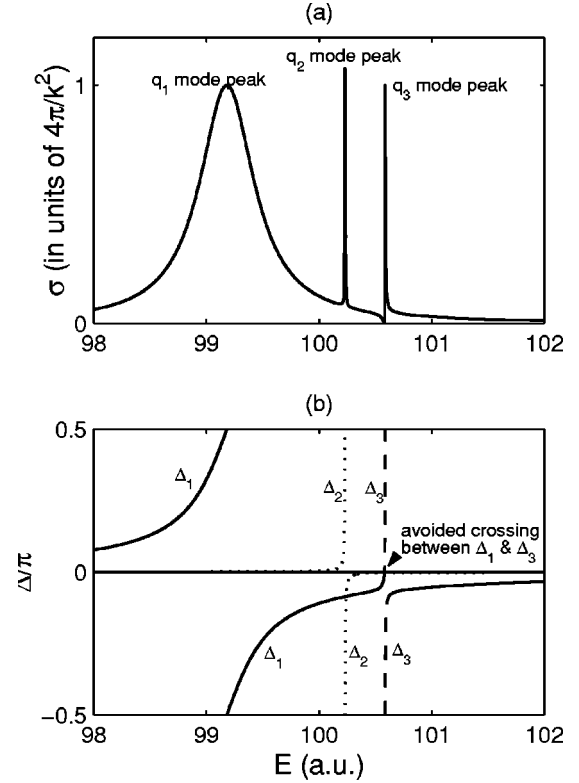


FIG. 2. (a) Total cross section  $\sigma$  and (b) phase angles  $\Delta_j$  of each eigenmode as a function of energy of a system of three identical scatterers placed on a straight line separated by equal distance  $d = 0.2/k_0$ , at  $(-d, 0, 0)$ ,  $(0, 0, 0)$ , and  $(d, 0, 0)$ . Each scatterer is the same as used in Fig. 1. In (b) solid lines,  $\Delta_1$ ; dotted lines,  $\Delta_2$ ; dashed lines,  $\Delta_3$ .

$$\vec{q}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{q}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (10)$$

corresponding to the two scatterers oscillating perfectly in and out of phase with the same strength. Equation (10) is valid at any energy. Figure 1 shows an example of the total cross section and phase angles as a function of energy. For  $kd < 1$ , i.e., the scatterers are placed within each other's effective radius, the "s-like" symmetric  $\vec{q}_1$  mode becomes a broad band, and the "p-like" antisymmetric  $\vec{q}_2$  mode becomes a sharp proximity resonance peak, both shifted from the peak of single scatterer, as shown in Fig. 1(a) [4]. In the limit of  $d \rightarrow 0$ ,  $\vec{q}_1$  mode rigorously becomes an s wave and  $\vec{q}_2$  mode a p wave.

The three-scatterer case turns out to have a richer phenomenology. We consider three identical scatterers positioned on a straight line separated by equal distance:  $\vec{r}_1 = (-d, 0, 0)$ ,  $\vec{r}_2 = (0, 0, 0)$ ,  $\vec{r}_3 = (d, 0, 0)$ . Compared to the two-scatterer case, the difference is that the relative amplitudes in each  $\vec{q}_j$  are now energy dependent. Nonetheless, upon reflection about  $y$ - $z$  plane, there are always two symmetric modes  $\vec{q}_1$ ,  $\vec{q}_3$ , and one antisymmetric mode  $\vec{q}_2$ . Figure 2(b) shows the phase angles as a function of energy, which determine

the positions and widths of the peaks in total cross section [Fig. 2(a)]. The eigenmodes at  $\vec{q}_1$  mode's peak ( $E = 99.182$ ) are

$$\vec{q}_1 = \begin{pmatrix} 0.089 \\ 0.106 \\ 0.089 \end{pmatrix}, \quad \vec{q}_2 = \begin{pmatrix} 1.23 \\ 0 \\ -1.23 \end{pmatrix}, \quad \vec{q}_3 = \begin{pmatrix} 23.9 \\ -47.5 \\ 23.9 \end{pmatrix}.$$

We see that  $\vec{q}_1$  is *s*-like,  $\vec{q}_2$  is *p*-like, and  $\vec{q}_3$  is *d*-like. Indeed, numerical evidence shows that at low energies ( $E < E_0$ )  $\vec{q}_1$ ,  $\vec{q}_2$ , and  $\vec{q}_3$  converge to *s*, *p*, and *d* waves when  $d \rightarrow 0$ . The  $\vec{q}_1$  mode gives most of the broadband and the other two modes give two sharp proximity resonance peaks.

If we disregard the continuity of  $\Delta_j(E)$ , it is determined up to an integral multiple of  $\pi$ . For reasons that will become clear in a moment, we have drawn Fig. 2(b) with restriction  $-\pi/2 < \Delta_j \leq \pi/2$ . At  $E \sim 100.578$ ,  $\Delta_1$  and  $\Delta_3$  undergo an avoided crossing: instead of one going sharply from some  $-\alpha$  to  $\pi - \alpha$  and the other remaining at 0,  $\Delta_1$  goes sharply from  $-\alpha$  to  $0^-$ , and  $\Delta_3$  goes sharply from  $0^+$  to  $\pi - \alpha$ . This is really an avoided crossing of eigenvalues  $\lambda_1$  and  $\lambda_3$  at  $\lambda \approx 0$ . Tracking  $\vec{q}_1$  and  $\vec{q}_3$  around the avoided crossing region, we find  $\vec{q}_1$  is *s*-like and  $\vec{q}_3$  is *d*-like before the crossing ( $E < 100.578$ ), yet  $\vec{q}_1$  becomes *d*-like and  $\vec{q}_3$  becomes *s*-like after the crossing ( $E > 100.578$ ). In contrast,  $\Delta_2$  rigorously crosses  $\Delta_1$  since it has a different symmetry. Indeed, if we remove the symmetry of the system, there is always avoided crossing whenever two phase angles tend to cross at 0 [Fig. 3(b)]. The  $\vec{q}_3$  mode peak exhibits a Fano line shape [Fig. 2(a)] (Ref. [14]). The dip around  $E \sim 100.578$  before the peak comes from the avoided crossing at 0.  $\text{Sin}^2 \Delta$  will exhibit a Fano line shape if  $\Delta$  goes from some  $-\alpha$  to  $\pi - \alpha$ , with a minimum at  $\Delta = 0$  and the peak still at  $\Delta = \pi/2$ . Therefore, there is always a dip when two phase angles tend to cross at 0. The deepness of the dip depends on how narrowly they avoid crossing.

Numerical evidence shows that for a general system of  $N$  identical scatterers confined well within each other's effective radius, there is always a broadband with resonance energy less than  $E_0$  and some sharp proximity resonance peaks based on the high-energy tail of the broadband (Fig. 4) [10]. At the peak, the broad band corresponds to a mode (say,  $\vec{q}_1$ ) in which all scatterers oscillate in phase, i.e., all  $q_{i1}$ 's have the same sign. The peak is broad because the waves scattered from each scatterer add up constructively. In this sense this is an *s*-like mode, corresponding to the superradiant state in scattering of one photon by a collection of atoms [8]. The remaining  $N-1$  modes give  $N-1$  sharp proximity resonance peaks (if they are separate). They correspond to sub-radiant states. The scattered waves of these modes at their respective peaks have little *s*-wave component—waves scattered from each scatterer tend to add up destructively. It has been reported that under certain conditions these narrow-peak eigenmodes correspond to Anderson localization [15].  $\vec{q}_1$  mode is *s*-like at low energies. As a consequence of avoided crossing of phase angles, the *s*-like character will

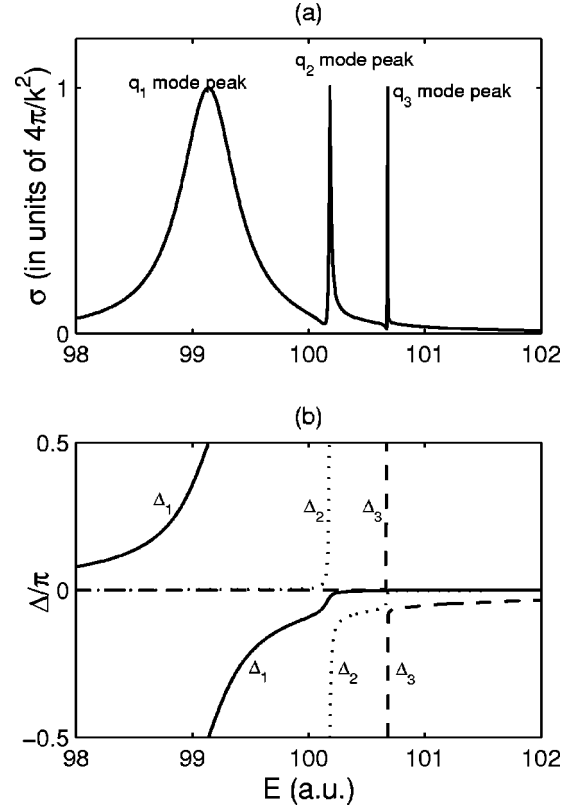


FIG. 3. (a) Total cross section  $\sigma$  and (b) phase angles  $\Delta_j$  of each eigenmode as a function of energy of a system of three identical scatterers placed at  $(-d, 0, 0)$ ,  $(-d/3, -d/3, 0)$ , and  $(d, 0, 0)$ , with  $d = 0.2/k_0$ . Each scatterer is the same as used in Fig. 1. In (b) solid lines,  $\Delta_1$ ; dotted lines,  $\Delta_2$ ; dashed lines,  $\Delta_3$ . The difference with Fig. 2 is that the scatterer at the center is moved so there is no longer symmetry about the reflection of *x-z* and *y-z* planes.

shift to different eigenmodes at different energies. Each sharp peak corresponds to a phase angle to sharply increasing from some  $-\alpha$  ( $0 < \alpha < \pi/2$ ) on the broadband (corresponding to *s*-like mode) to  $\pi - \alpha$ —back onto the broadband with avoided crossing at 0, exhibiting a Fano line shape. We may not see the dip if the phase angles do not come close enough in avoided crossing, but in principle a sharp proximity resonance peak is always unsymmetric [Figs. 2(a) ( $\vec{q}_3$  peak only), 3(a), and 4(a)].

It is surprising to find numerically that, even for a typical nonsymmetric system of identical scatterers, at far off-resonant energies while still maintaining  $kr_{ij} \ll 1$ ,  $y_j(\hat{r})$ 's are very much like linear combinations of spherical harmonics of same *l*, i.e., they almost have well-defined angular momenta.  $y_j(\hat{r})$ 's also have lowest *l*'s possible, so they are still *s*-, *p*-, *d*-, . . . like. At resonant energies, the eigenmodes have mixed *l*'s in avoided crossing region.

## V. SUMMARY

We have shown that for a system of  $N$  *s*-wave point scatterers, we can always find  $N$  eigenmodes. These eigenmodes or eigenchannels play the same role as spherical harmonics for a spherically symmetric target. The *T* matrix of the system is

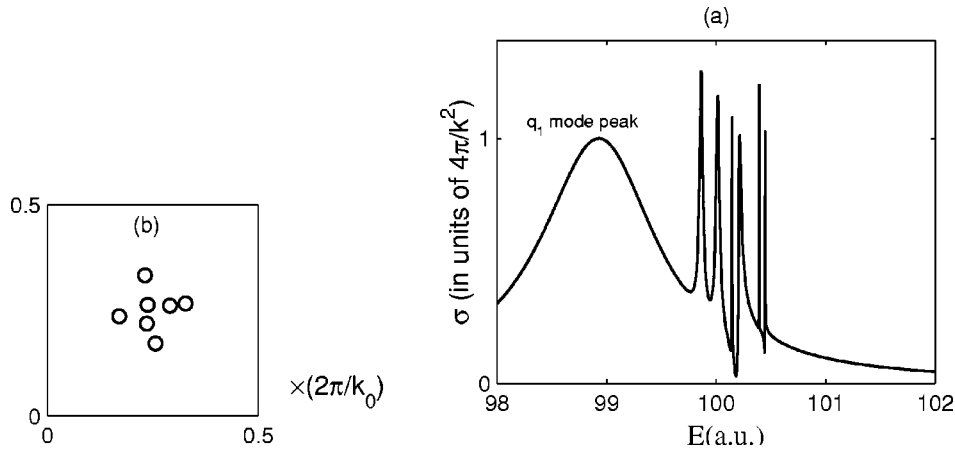


FIG. 4. (a) Total cross section as a function of energy for a system of seven identical scatterers randomly placed on a plane. Each scatterer is the same as used in Fig. 1. The positions of the scatterers are shown in (b).

of rank  $N$  and can be represented analytically. The eigenmodes are eigenvectors corresponding to nonzero eigenvalues of the  $T$  matrix. The position and width of resonance peaks are determined by phase angles as a function of energy. Each phase angle monotonically increases from 0 to  $\pi$  as the energy goes from  $-\infty$  to  $\infty$  if each scatterer has one Breit-Wigner-type resonance.

For identical scatterers placed well within one wavelength of the incoming wave, the far off-resonant eigenmodes are always  $s$ -,  $p$ -,  $d$ -, . . . like with lowest possible angular momenta. At resonant energies where phase angles have avoided crossings, the eigenmodes have components of different  $l$ 's mixed. There is always one broadband originating from an  $s$ -like mode whose peak is shifted lower in energy, and  $N-1$  sharp proximity resonance peaks based on the high-energy tail of the broadband. The broadband peak corresponds to superradiant state in scattering of light, and the sharp peaks corresponds to subradiant states. The unsymmetric Fano line shape of a sharp peak comes from a phase angle

sharply increasing from some  $-\alpha$  on the broad band to  $\pi - \alpha$  (back onto the broadband) with avoided crossing at 0.

In the future, we will investigate the generalization of eigenmode expansion to systems of multichannel point scatterers. We will investigate further the dependence of the spectrum (total cross section as a function of energy) on the spatial distribution of scatterers, and statistical properties of the spectra of collections of a large number of randomly distributed scatterers. We need to extend the theory to non-identical scatterers, and to multiphoton emission of collections of atoms (i.e., true superradiance).

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